Potential Flow

For a 2D flow region in which viscous forces are negligible, irrotationality may be a valid assumption. The potential function, ϕ is used for defining the velocity field in an irrotational flow. Due to this, the irrotational flow theory is often referred to as potential flow theory.

Although all real fluids possess viscosity, but there are many situations in which the assumption of inviscid flow considerably simplifies the analysis, and yet provides useful results.

Laplace's Equation:

Mass conservation eq. for an incompressible flow can be written as.

$$\vec{\nabla} \cdot \vec{V} = 0$$

For an irrotational flow we can define a velocity potential function, such that

$$\vec{V} = \vec{\nabla}\phi$$

$\therefore$ For, an incompressible, irrotational flow

$$\bar{\nabla} \cdot (\bar{\nabla}\phi) = 0$$

or $\nabla^2\phi = 0$ – Laplace's eq. [Read about him]

Note: Solutions for Laplace's eq. are called harmonic fn.

P.S: See Laplace's eq in cyl. & sph. coordinates.

We also know, for incompressible flow

$$u = \frac{\partial \psi}{\partial y} \quad \& \quad v = -\frac{\partial \psi}{\partial x}$$

For irrotational flow,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

or $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$ – Again Laplace eq.

Note:

1. Any soln. of Laplace's eq. represents a velocity potential or stream function (which is 2D). for an irrotational, incompressible flow.

2. Laplace's eq. is second-order linear PDE.

3. Sum of any particular solutions of a linear DE is also the soln. of the eq.

   If $\phi_1, \phi_2, \cdots \phi_n$ are n separate soln of Laplace eq.

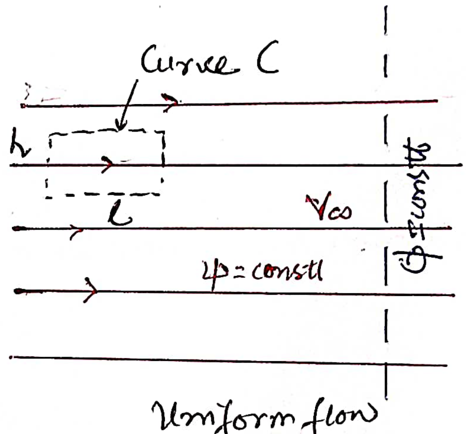
   then $\phi = \phi_1 + \phi_2 + \cdots + \phi_n$ is also a soln.

4. We can say that since irrotational, incompressible flow is governed by Laplace's eq. A complicated flow pattern for an ideal flow can be represented by adding together a number of elementary flow soln. that are also ideal.

Consider ideal flow over different shapes like a sphere, an aerofoil etc. Each flow is different. But they all are governed by $\nabla^2\phi=0$. How do we obtain these different flows, by using different B.C.

Now, Let us consider different elementary flows.

Uniform flow.



Consider a uniform flow with velo V_∞ oriented in +ve x dir.

A uniform flow is a physically possible incompressible ($\bar\nabla\cdot\bar V=0$) and irrotational ($\bar\nabla\times\bar V=0$) flow.

A velo. potential for uniform flow can be obtained such that $\vec\nabla\phi=\vec V$.

1 - $\dfrac{\partial\phi}{\partial x}=u=V_\infty$ — (1)

2 - $\dfrac{\partial\phi}{\partial y}=v=0$

Integrating above eq^n (1) w.r.t x, we have.

$$\phi=V_\infty x+f(y) \quad -3$$

where $f(y)$ is a fn of y only.

Integrating eq.(2) w.r.t. y.

$$\phi=\text{const}+g(x) \quad -(4) \qquad g\equiv g(x)$$

Comparing (3) & (4)

$$\phi=V_\infty x+\text{constt} \quad -(5)$$

Since, we are generally interested in velocity, & $\vec\nabla\phi=\vec V$ & derivative of constt is zero, ∴ we can say $\phi=V_\infty x$ — (6)

For the uniform flow, considered before.

$$\frac{\partial \psi}{\partial y} = u = V_\infty \qquad -(7)$$

$$\frac{\partial \psi}{\partial x} = -v = 0 \qquad -(8)$$

Integrating eq (7) w.r.t y & eq (8) w.r.t x, and comparing

we have $\quad \psi = V_\infty y \qquad -(9)$

Circulation:

$$\Gamma \equiv -\oint_c \bar{V}\cdot d\bar{s}$$

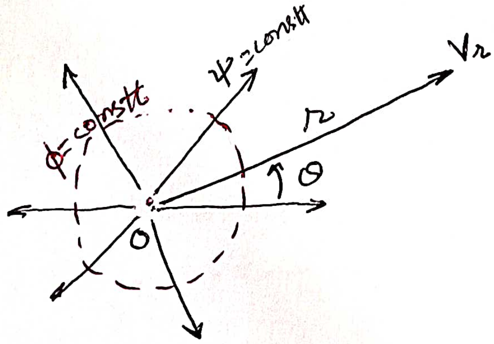
$$\Gamma = -\bar{V}_\infty \cdot \oint d\bar{s} = V_\infty \cdot 0 = 0$$

Line integral of $d\bar{s}$ around a closed curve is identically zero.

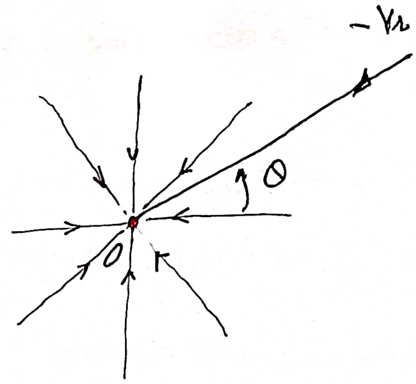
Also, $\Gamma = -\int_s (\bar{\nabla} \times \bar{V}) \cdot ds$

For irrotational flow $\bar{\nabla} \times \bar{V} = 0 \quad \therefore \Gamma = 0.$

Source flow



Source flow



Sink flow

Consider a 2D, incompressible flow, where all streamlines are straight lines emanting from a central point, O. The velocity along each streamline vary inversely with distance from O. Let The radial velocity be V_r, & tangential velo. be V_θ. where $V_\theta = 0$.

From incompressibility

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r}\left(\frac{\partial(rV_r)}{\partial r}\right) + \frac{1}{r}\cancelto{0}{\frac{\partial V_\theta}{\partial\theta}} = 0$$

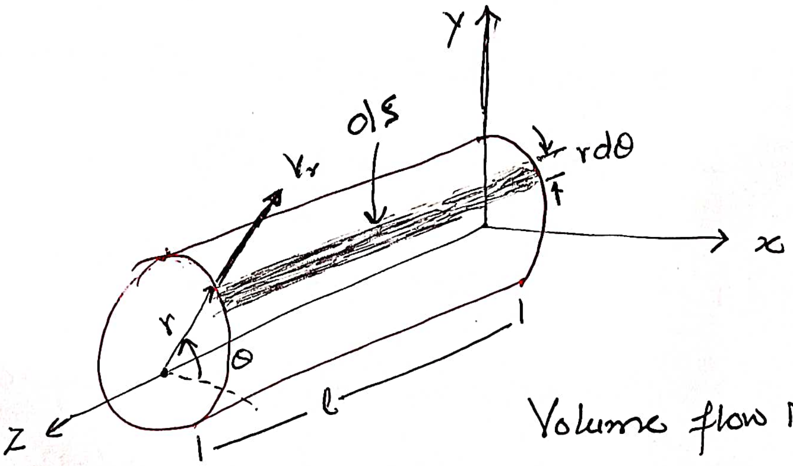
$$\Rightarrow \quad \frac{\partial}{\partial r}(rV_r) = 0$$

or, $\quad V_r = \dfrac{c}{r}$; after integrating.

Note: $\quad c>0$; Source flow

$\qquad\qquad c<0$; Sink flow.

The value of c is related to the volume flow from the source.



Volume flow rate from a line source.

The 2D flow shown ~~before~~ above is the same in any plane $\perp$ z axis. Consider the mass flow across the surface of the cylinder of radius r and height l. The elemental mass flow across the surface element ds is $\rho\bar{V}\cdot d\bar{s} = \rho V_r (r d\theta)(l)$. Since V_r is same at any θ for fixed r. The total mass flow across the surface of the cylinder is

$$\dot{m} = \int_0^{2\pi} \rho V_r (r d\theta)\, l = \rho r l V_r \int_0^{2\pi} d\theta = 2\pi r l \rho V_r$$

or $\quad \dfrac{\dot{m}}{\rho} = \dot{\forall} = 2\pi r l V_r$

or $\quad \Lambda = \dfrac{\dot{\forall}}{l} = 2\pi r V_r$

Or $V_r = \frac{\Lambda}{2\pi r}$ capital lambda

Comparing with $V_r = \frac{C}{r}$

$C = \Lambda / 2\pi$

Note: Λ defines the source length; it is physically the rate of Volume flow from the source, per unit depth perpendicular to the page, $\Lambda \rightarrow m^2/s$

Also, $\frac{\partial \phi}{\partial r} = V_r = \frac{\Lambda}{2\pi r}$ — (A)

& $\frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_\theta = 0$ — (B)

Integrating (A) w.r.t r, & (B) w.r.t θ, we get

$\phi = \frac{\Lambda}{2\pi} \ln r + f(\theta)$ — (C)

& $\phi = \text{constt} + f(r)$ — (D)

Comparing (C) & (D) we see that.

$f(r) = (\Lambda/2\pi) \ln r$; $f(\theta) = \text{constt}$.

$\therefore \phi = \frac{\Lambda}{2\pi} \ln r.$

Note: The constt can be dropped. Since we are interested in Velo. which is derivative of ϕ. Derivative of a constt is zero.

Similarly for stream function.

$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_r = \frac{\Lambda}{2\pi r}$; $-\frac{\partial \psi}{\partial r} = V_\theta = 0$

Int. w.r.t. θ — Int. w.r.t. r

$\psi = \frac{\Lambda}{2\pi}\theta + f(r)$ — $\psi = \text{constt} + f(\theta)$

$\psi = \frac{\Lambda}{2\pi}\theta$

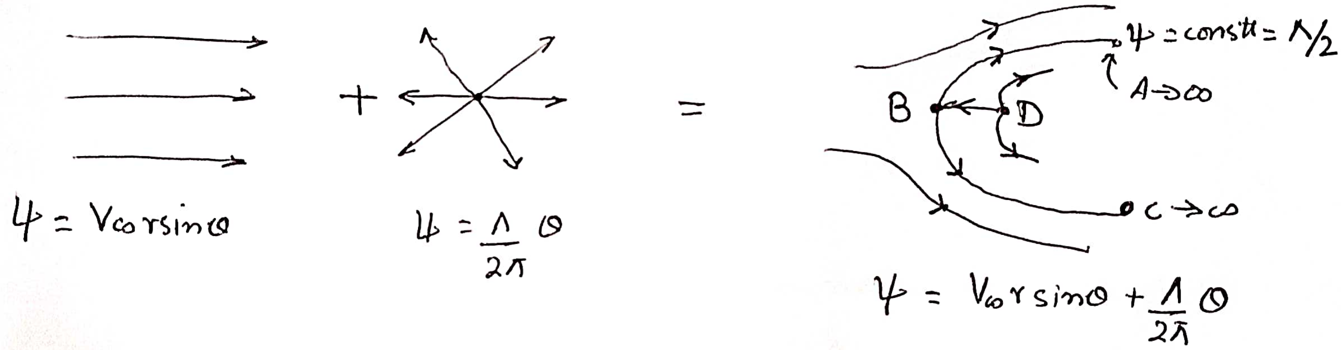
P.S: 1. How streamlines are radial straight lines from origin

2. How equipotential lines are circles with center at origin

3. How streamlines & equipotential lines are mutually perpendicular

4. Circulation for source flow.

Combination of Uniform flow with a Source.



Consider a source of strength Λ, located at the origin. Superimpose on this a uniform flow with velocity V_∞.

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta . \quad , \text{Assume } k = \frac{\Lambda}{2\pi}$$

Soln. of Laplace eq.

$\therefore \psi$ is soln. of Laplace eq.

The velocity field is obtained by.

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta + \frac{k}{r}$$

$$\& \quad V_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta$$

$\rightarrow$ (A)

Note: V_r for source flow is $\frac{\Lambda}{2\pi r}$, & the comp of free stream velo in radial direction is $V_\infty \cos\theta$. Thus, the V_r obtained above for the (Source + Uniform) flow is simply the sum of the two velocity fields.

2. This is consistent with the linear nature of the Laplace's eq.

The stagnation points in the flow can be obtained by setting V_r & V_θ to zero.

$$V_\infty \cos\theta + \frac{\Lambda}{2\pi r} = 0$$

$$V_\infty \sin\theta = 0$$

Solving for r & θ, we get. $(r,\theta) = (\Lambda/(2\pi V_\infty), \pi)$

Note: 1. Stagnation pt is $(\Lambda/2\pi V_\infty)$ directly upstream of source.

2. DB increases if Λ increases or $V_\infty \downarrow$.

Substituting the stagnation pt. coordinates into the ψ eq.

$$\psi = V_\infty \frac{\Lambda}{2\pi V_\infty} \sin\pi + \frac{\Lambda}{2\pi}\pi = \text{const.}$$

or $\psi_{stag} = \frac{\Lambda}{2} = \text{const.}$ → represents curve ABC.

Note: 1. In inviscid flow, velo at body surface is tangent to the body. Therefore, any stream of the combined flow (half body) could be replaced by a solid surface of the same shape.

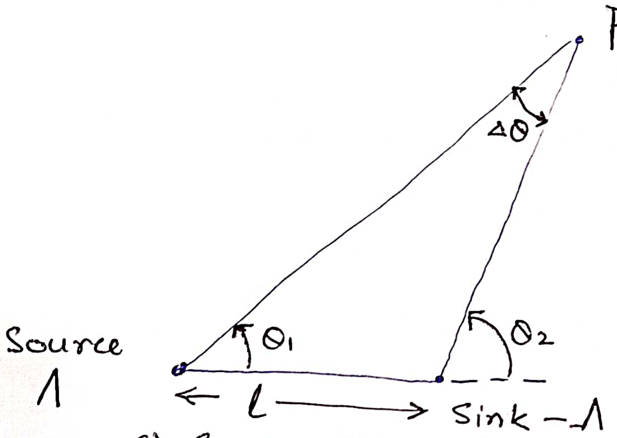
2. Streamline ABC acts as a dividing streamline. It separates fluid coming from free stream & emanating from the source.

3. The flow discussed before represents the flow over solid semi-infinite body ABC.

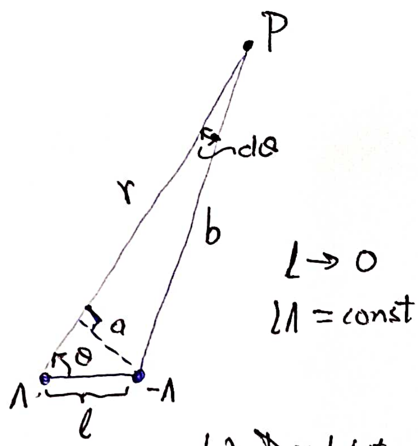
P.S: See Rankine oval.; Uniform + [Source + Sink].
Kelvin oval

Doublet flow.

Consider a source of strength Λ & sink of strength $-\Lambda$, separated by a distance l.



a) Source-sink pair



b) Doublet.

At any point P in the flow, the stream function is.

$$\psi = \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = -\frac{\Lambda}{2\pi}\Delta\theta \quad ; \quad \Delta\theta = \theta_2 - \theta_1$$

Now, Let 'l' approach zero in such a way that '$l\Lambda$' remains constt.

In this Limit of $l \to 0$ & $l\Lambda = \text{constt}$, we obtain a special flow pattern Known as doublet.

The strength of doublet is $l\Lambda \equiv k$

& $$\psi = \underset{\substack{l \to 0 \\ k = l\Lambda = \text{constt}}}{Lt} \left(-\frac{\Lambda}{2\pi} d\theta\right)$$

In the Limit $\Delta\theta \to d\theta \to 0$ & $\Lambda \to \infty$,

From fig. (b).

$$a = l\sin\theta$$
$$b = r - l\cos\theta$$
$$d\theta = a/b = \frac{l\sin\theta}{r - l\cos\theta}$$

$$\psi = \underset{\substack{l \to 0 \\ k = \text{constt}}}{Lt} \left(-\frac{\Lambda}{2\pi}\frac{l\sin\theta}{r - l\cos\theta}\right)$$

or $\psi = \lim_{\substack{l \to 0 \\ k = \text{constt}}} \left(-\frac{k}{2\pi} \frac{\sin\theta}{r - l\cos\theta} \right)$

$$\psi = -\frac{k}{2\pi} \frac{\sin\theta}{r}$$

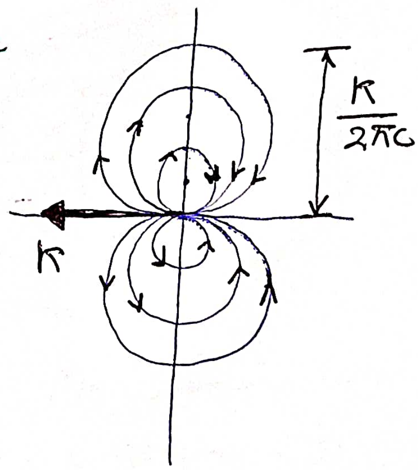
Similarly, $\phi = \frac{k}{2\pi} \frac{\cos\theta}{r}$ → P.S: Derive it

To obtain the stream lines of a doublet.

$$\psi = -\frac{k}{2\pi} \frac{\sin\theta}{r} = \text{constt} = c$$

$$\text{or } r = -\frac{k}{2\pi c} \sin\theta$$

Comparing above eq. with $r = d\sin\theta$. and recalling from analytical geometry that $r = d\sin\theta$ in polar coordinates is a circle with a diameter 'd' on the vertical axis and with center located '$d/2$' directly above origin.



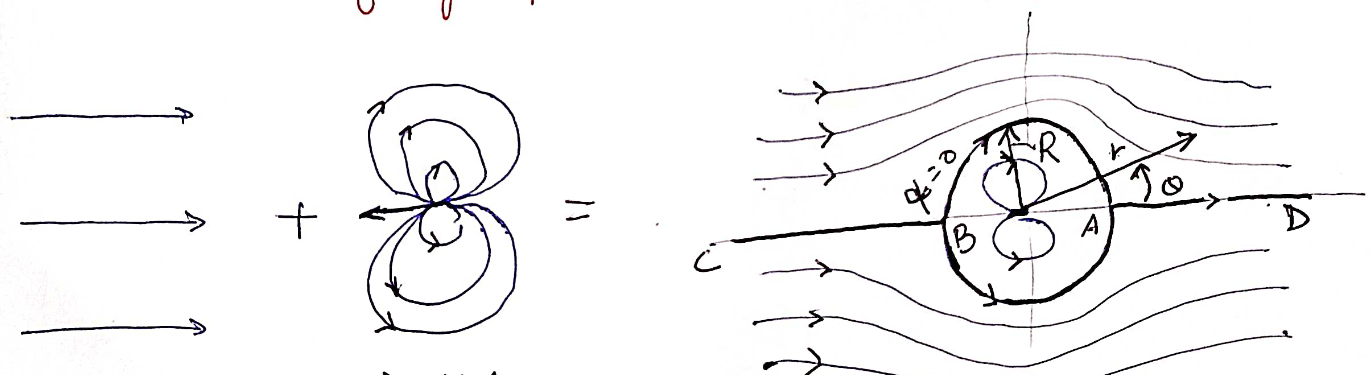
Stream lines for a doublet are a family of circles with dia. $k/2\pi c$.

- Different circles for different value of c.
- Due to the placing of sink to the right of source, the direction of flow is out of the origin to left. If we interchange the source & the sink locations, the flow dir. would change.
- Therefore, we can say that a doublet has a sense of direction associated with it. By convention, the direction is shown by an arrow going from sink to source.
- For the limit $l \to 0$, source & sink fall on top of each other.

- They do not cancel each other, as the absolute magnitude of their strengths become infinitely large in the limit and we have a singularity of strength ($\infty - \infty$); this is an indeterminate form, which can have a finite value.

- It is helpful to interpret the doublet as being induced by a discrete doublet of strength κ placed at the origin. Doublet is a singularity that induces about it the double lobed circular flow pattern.

Nonlifting flow over a circular cylinder



Uniform flow
$\psi = V_\infty r \sin\theta$

Doublet
$\psi = \frac{-\kappa}{2\pi} \frac{\sin\theta}{r}$

Flow over a cylinder
$\psi = V_\infty r \sin\theta - \frac{\kappa}{2\pi} \frac{\sin\theta}{r}$

or $\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)$; $R = \sqrt{\frac{\kappa}{2\pi V_\infty}}$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}(V_\infty r\cos\theta)\left(1 - \frac{R^2}{r^2}\right)$$

$$= \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos\theta.$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -\left[(V_\infty r\sin\theta)\frac{2R^2}{r^3} + \left(1 - \frac{R^2}{r^2}\right)(V_\infty \sin\theta)\right]$$

$$= -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin\theta.$$

For locating stagnation points, we put $V_r = 0$ & $V_\theta = 0$

$$\left(1 - \frac{R^2}{r^2}\right) V_\infty \cos\theta = 0$$

$$\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin\theta = 0$$

Simultaneously solving both equations for r & θ, we find two stagnation points, located at $(r, \theta) = (R, 0)$ & (R, π). These points are denoted as A and B.

By putting coordinates of A & B in ψ we find $\psi = 0$. This implies that same streamlines passes through A & B. The eq. of this stream line is

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) = 0$$

Since $\theta = \pi$ & $\theta = 0$ satisfies the above eq. hence the entire horizontal axis extending infinitely far upstream of B and downstream of A, is part of the stagnation streamline.

Note:

1. Entire flow field is symmetrical about both the horizontal and the vertical axes. Hence the pressure distribution is also symmetrical. Thus there is no net lift and no drag.

2. Zero lift is easy to digest, but zero drag in real life makes no sense.

3. This paradox between theoretical & actual drag was first observed by Jean Le Rond d'Alembert in 1744 and hence known as d'Alembert's paradox.

4. During 18th & 19th centuries this paradox was unexplained by the researchers.

5. Now we know that drag is due to viscous effects which generate frictional shear stress at the surface of the body. this causes the flow to separate from the surface on the back of the body, thus creating a wake and destroying the symmetry of the flow about the vertical axis.

The velo. distribution on cylinder surface, $r = R$ is.

$V_r = 0$; velo. ⊥ solid surface $= 0$.

$V_\theta = -2V_\infty \sin\theta$. ; full magnitude of velo. $V = V_\theta$.

-. sign is due to the sign convention V_θ is +ve. counterclockwise.

V_θ is max. $(= 2V_\infty)$ at the top $(\theta = \pi/2)$ and bottom $(\theta = 3\pi/2)$.

Applying Bernoulli's eq. between a point in free stream and a point on the cylinder.

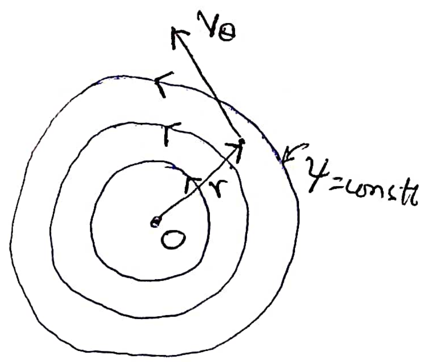
$$\frac{P}{\rho g} - \frac{P_\infty}{\rho g} = \frac{V_\infty^2}{2g} - \frac{V^2}{2g}$$

or $$\frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \left(\frac{V}{V_\infty}\right)^2 = C_p \text{ ; pressure coefficient.}$$

or $$C_p = 1 - 4\sin^2\theta.$$

P.S: Cp distribution over the entire cylinder surface.

Vortex flow.



Vortex flow

- Fluid particles moves in circles about a point.
- $V_r = 0$ and V_θ varies with r.
- All streamlines are concentric circles where V_θ is inversely proportional to r.

In a purely circulatory (free vortex flow) motion, the tangential velocity can be written as.

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r}\left(\frac{\partial}{\partial r}\cancel{(rV_r)}^{0}\right) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$$

Thus for an incompressible flow,

$$\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \Rightarrow V_\theta \neq f(\theta)$$

Also, $\bar{\nabla}\times\bar{V} = 0 \Rightarrow \frac{\partial V_\theta}{\partial r} + \frac{v_\theta}{r} = 0$

$$\text{or} \quad \frac{dv_\theta}{v_\theta} = -\frac{dr}{r}$$

$$\text{or} \quad \ln V_\theta = -\ln r + \ln c$$

$$\text{or} \quad V_\theta = \frac{c}{r}$$

For finding c, we take Γ around a streamline of r.

$$\Gamma = -\oint_c \bar{V}\cdot d\bar{s} = -V_\theta(2\pi r)$$

$$V_\theta = -\frac{\Gamma}{2\pi r}$$

$$\Rightarrow C = -\frac{\Gamma}{2\pi}$$

For vortex flow, circulation taken about all streamlines is the same, $\Gamma = -2\pi C$.

Γ is the strength of the vortex flow.

P.S: Why vortex flow is irrotational except at the origin.
What happens at $r=0$.
Find $\bar{\nabla}\times\bar{V}$ at $r=0$.
Use $\Gamma = -\iint(\bar{\nabla}\times\bar{V})\cdot d\bar{S}$ where $\bar{S}$ is area

Now,

$$\frac{\partial\phi}{\partial r} = V_r = 0 \quad ; \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = V_\theta = -\frac{\Gamma}{2\pi r}$$

Integrating above eq. we find,

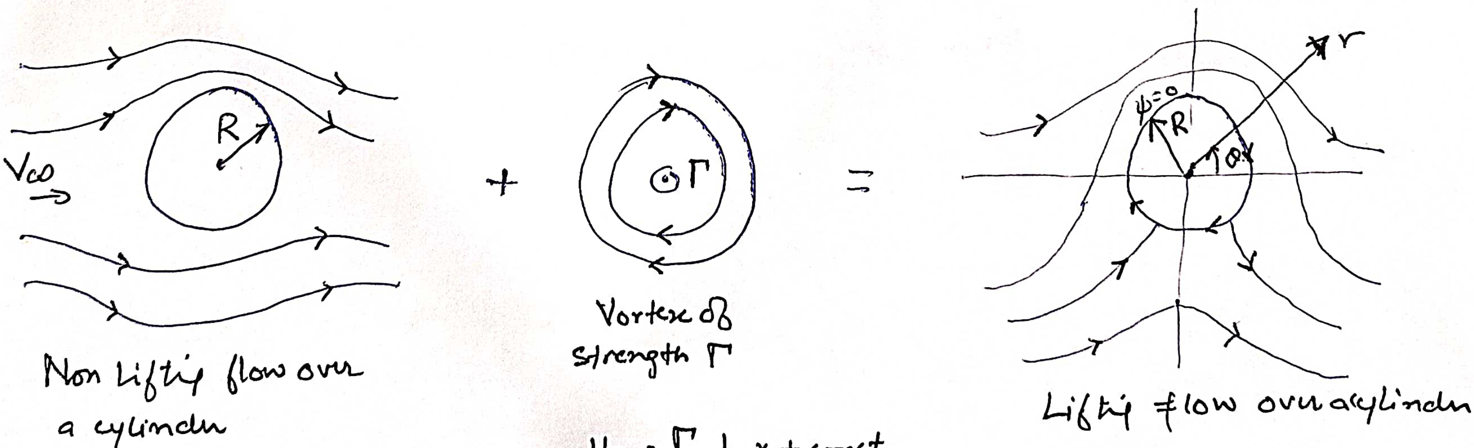
$$\phi = -\frac{\Gamma}{2\pi}\theta$$

Similarly.

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_r = 0 \quad ; \quad -\frac{\partial\psi}{\partial r} = V_\theta = -\frac{\Gamma}{2\pi r}$$

$$\psi = \frac{\Gamma}{2\pi}\ln r$$

Lifting flow over a cylinder.



$$\psi_1 = (V_\infty r\sin\theta)\left(1-\frac{R^2}{r^2}\right)$$

$$\psi_2 = \frac{\Gamma}{2\pi}\ln r + \text{const}$$

Since the const is arbitrary let it be, $\text{Const} = -\frac{\Gamma}{2\pi}\ln R$

$$\therefore \psi_2 = \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

$$\psi_{lifting} = (V_\infty r\sin\theta)\left(1-\frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

- Streamlines are symmetrical about vertical axis (no drag) but are unsymmetrical about horizontal axis (Lift exists)
- Because a vortex of strength Γ has been added to the flow, the circulation about the cylinder is finite ($=\Gamma$).

Now, we can find the velocities by adding the component flow velocities.

$$\therefore \quad V_r = \left(1-\frac{R^2}{r^2}\right)V_\infty\cos\theta \quad \text{— } 1$$

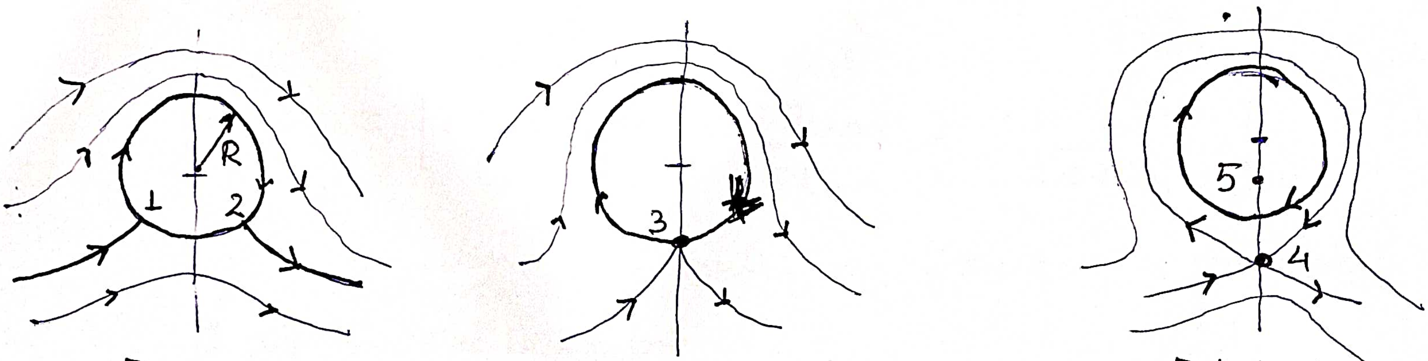
$$V_\theta = -\left(1+\frac{R^2}{r^2}\right)V_\infty\sin\theta - \frac{\Gamma}{2\pi r} \quad \text{— } (2)$$

For locating the stagnation points put $V_r = V_\theta = 0$

From eq (1) $r = R$. Substituting $r = R$ in eq (2). we get

$$\theta = \arcsin\left(-\frac{\Gamma}{4\pi V_\infty R}\right) \quad \text{— } (3)$$

Since Γ is +ve, $\therefore$ θ must be in the 3rd & 4th quadrant.



a) $\Gamma < 4\pi V_\infty R$ b) $\Gamma = 4\pi V_\infty R$ c) $\Gamma > 4\pi V_\infty R$

The two stagnation points (1, 2) are shown in fig a). for $r=R$ and θ as given by eq (3).

Above result is valid only when $\Gamma/4\pi V_\infty R < 1$

If $\Gamma/4\pi V_\infty R > 1$, then eq(3) has no meaning. If $\Gamma/4\pi V_\infty R = 1$ there is only one stagnation point on the cylinder surface $(R, -\pi/2)$. as shown in fig(b).

For $\Gamma/4\pi V_\infty R > 1$, if we see eq(1) then we find that it is also satisfied for $\theta = \pi/2$ or $-\pi/2$.

Substituting $\theta = -\pi/2$ into eq(2), we get.

$$r = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{\left(\frac{\Gamma}{4\pi V_\infty}\right)^2 - R^2}$$

So, we have 2 stagnation pts. one inside and the other outside the cylinder.

Calculation of Cp

Velocity on the surface of the cylinder

$$V = V_\theta = -2V_\infty \sin\theta - \frac{\Gamma}{2\pi R}$$

As obtained for the case of non lifting cylinder

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \left(-2\sin\theta - \frac{\Gamma}{2\pi R V_\infty}\right)^2$$

$$= \left[1 - 4\sin^2\theta \mp \frac{2\Gamma \sin\theta}{\pi R V_\infty} \mp \left(\frac{\Gamma}{2\pi R V_\infty}\right)^2\right]$$

Using Bernoulli eq. the surface pressure P & P_∞ are related by.

$$P_\infty + \frac{1}{2}\rho V_\infty^2 = P + \frac{1}{2}\rho\left(-2V_\infty \sin\theta - \frac{\Gamma}{2\pi R}\right)^2$$

or $$P - P_\infty = \frac{1}{2}\rho V_\infty^2\left(1 - 4\sin^2\theta + \frac{2\Gamma}{\pi R V_\infty R}\sin\theta - \frac{\Gamma^2}{4\pi^2 V_\infty^2 R^2}\right)$$

Drag & Lift forces.

Drag: $D = -\int_0^{2\pi} (p-p_\infty) \cos\theta \, R d\theta$; per unit depth

$= 0$

P.S: Derive that.

Lift: $L = -\int_0^{2\pi} (p-p_\infty) \sin\theta \, R d\theta$. (taken +ve upwards)

$$= -\frac{1}{2} \rho V_\infty^2 \frac{4K}{R V_\infty} R \int_0^{2\pi} \sin^2\theta \, d\theta$$

[Since the integral of any odd power of $\sin\theta$ over 2π is zero.]

where $K = \frac{\Gamma}{2\pi}$.

$L = -\rho V_\infty \Gamma$ (Kutta-Joukowski theorem)